STRONGLY W FLAT MODULES OVER MATLIS DOMAIN

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Definition: Let R be an arbitrary ring. We will say that a module $M \in R-$mod is w-flat if the functor $-\otimes_R(M_1+M_2)$ is exact on the category R-module. In other words, if whenever $O \to A \to B \to C \to O$ is a short exact sequence in R-mod. the

$O \to A \otimes_R (M_1+M_2) \to B \otimes_R (M_1+M_2) \to C \otimes_R (M_1+M_2) \to O$ is a short exact sequence. Since $-\otimes_R(M_1+M_2)$ is exact on the R.H.S., the module M is w-flat if any $A \otimes_R (M_1+M_2) \to B \otimes_R (M_1+M_2)$.

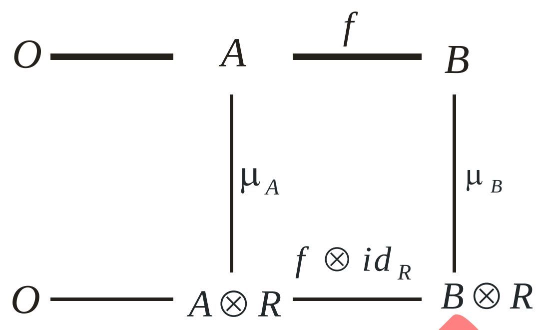
Lemma 1 : Let R be a ring with local units. For any ME mod-R the map $\mu(M_1+M_2):(M_1+M_2)$ $\to (M_1+M_2)$ given by $\sum_{\substack{i-1\\j=1}}^{n}(M_i+M_j)\otimes r_i \to \sum_{\substack{i-1\\j=1}}^{n}(M_i+M_j)\otimes r$ is an isomorphism of right R-modules.

Proof : Since $(M_1+M_2)R=(M_1+M_2)$this map is clearly an epimorphism. Suppose $\sum_{\substack{i-1\\j=1}}^{n}(M_i+M_j)r_i = O.$ Let $\in$be a local unit in R-Satisfying $er_i = r_i e = r_i$ for i=1, 2, 3……n. Then

$$\sum_{\substack{i-1\\j=1}}^{n}(M_i+M_j)\otimes r_i = \sum_{\substack{i-1\\j=1}}^{n}(M_i+M_j)\otimes r_i \in = \sum_{\substack{i-1\\j=1}}^{n}(m_i r_i + m_j r_i)\otimes \in = \otimes \in = O.$$

Corollary : A ring R with local units is w- flat as a left R-module.

Proof: Let $A = A_1 + A_2$ and $B = B_1 + B_2$ .let $O \longrightarrow A \xrightarrow{f} B$ be an exact sequence of right R-modules. Tensoring with the left R-module R, we get a commutative diagram.



Where id_R is the identity map on R and μ_A, μ_B are the isomorphism defined in lemma 1. Since μ_A, f and μ_B are all monomorphism, so is $f \otimes id_R$. Hence R is w flat in R-Mod.

Proposition 3 : Let A is an ideal of a ring R, The following condition's are equivalent:

1. $A = (A_1 + A_2)$ is a pure ideal of R.
2. For each finite family $\alpha_i \le i \le n$ of elements of A there exists $t \in A$ such that $\alpha_i = \alpha_i t \forall i, l \le i \le n$.
3. For all $\alpha \in A$ there exists $\beta \in A$ such the $\alpha = \alpha\beta$.
4. $\frac{R}{A_1 + A_2}$ is a w-flat R-module.

Morever, if A is finitely generated, then A is pure if and only if it is generated by an idempotent.

Proof: (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv) Let B be an ideal of R. we must prove that $A \cap B = A.B.$ If $a \in (A_1 + A_2) \cap B$ there exists $t \in (A_1 + A_2)$ such that $\alpha = \alpha t.$ Hence $\alpha \in (A_1 + A_2)B.$

(iv) $\Rightarrow$ (iii) If $\alpha \in A_1 + A_2$, then $R\alpha = (A_1 + A_2) \cap R\alpha = (A_1 + A_2)\alpha = A\alpha$.

(i) $\Rightarrow$ (iii) If $\alpha \in A_1 + A_2$, 1 is solution of the equation $\alpha x = \alpha.$ So this equation has a solution in $A_1 + A_2 = A.$

(iii) $\Rightarrow$(ii) Let $\alpha_1, \alpha_2, \alpha_3 ....\alpha_n$ be elents of $(A_1 + A_2) = A$ we proceed by induction on n. There exists $t \in (A_1 + A_2)$ such that $\alpha_m = t\alpha_m$. By induction hypothesis there exists $S \in A_1 + A_2$ Such that $\alpha_n - t\alpha_n = s(\alpha_n - t\alpha_n) \forall i.l \le i \le (n-1).$

Now, it is easy to check that $(s + t - st)\alpha_i = \alpha_i \forall i, l \le i \le n.$

(ii) $\Rightarrow$(i) we consider the following system of equation's $\sum_{i-1}^{n} r_{j.i} r_i = \alpha_j \in A, I \leq J \leq P.$

Assume that $(\beta_1, \beta_2, \beta_3......\beta_n)$ is a solution of this system in R. There exists $S \in (A_1 + A_2)$ such that $\alpha_j = S\alpha_j \forall j, I \leq J \leq P,$ so $(s\alpha_1, s\alpha_2, s\alpha_3..........s\alpha_n)$ is a solution of this system in $A_1, A_2 = A$.

Lemma : Let $O \to A \to B \to C \to O$ be an exact sequence such that A and C are strongly w-Flat modules, then B is strongly w-Flat modules.

Proof: Let M be a strongly w-flat modules, by induced exact sequence.

$Ext_R^1(C \otimes (M_1 + M_2)) \to Ext_R^1(B \otimes (M_1 + M_2)) \to Ext_R^1(A \otimes (M_1 + M_2))$

Singe $Ext_R^1(C \otimes (M_1 + M_2)) = O$ and $Ext_R^1(A \otimes (M_1 + M_2)) = O$ then $Ext_R^1(B \otimes (M_1 + M_2)) = O$

Example : The Z-module Q is strongly Flate Modules Recall that R is called a mantles domain if the projective dimension of Q (or, equivalently, k) is one. The module C is called matlis cotorsion if $Ext_R^1(Q \otimes C) = O$ and M is called strongly w-Flat if $Ext_R^1((M_1 + M_2) \otimes C) = O$ for every Matlis cotorsion R-Module C.

Corollary : Let R is a semi-Dedkind domain. If M is a w-projective R-module and N is weak w-Projective R-Module then $(M_1 + M_2) \otimes_R {}^N$ is weak w-Projective.

Proof: The is omorphism $Tor_n^R((M_1 + M_2) \otimes N, A) \cong (M_1 + M_2) \otimes Tor_n^R(N, A)$ together pure w-projective and pure w-injectve

It follows $\sigma(M_1 + M_2) : (M_1 + M_2) \to E(M_1 + M_2)$ denotes the pure injective envelope of an R-module M where $M = M_1 + M_2$. Recall that an ijective envelope $\sigma(M_1 + M_2) : (M_1 + M_2) \to E(M_1 + M_2)$ has the unique mapping property (1) if for and homomorphism f: $(M_1 + M_2) \to N$ with kerf = O and N is pure injective, there exist a unique homomorphism g: $E(M_1 + M_2) \to E$ such that $g\sigma m : f$.where $M_1 + M_2 = M$.

Theorem : The following statements are equivalent.

(a) R is a priifer domain

(b) Every R-module is pure w-projective.

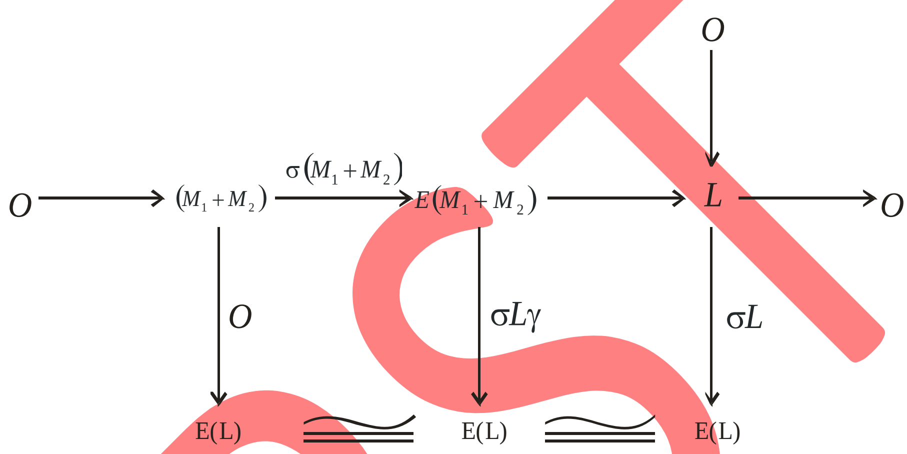
(c) $Ext_R^{\,1}(M_1 + M_2, N) = O$ for all pure w-injective R-module N.

(d) Every pure w-injective R-module has on injective envelope with the unique mapping property.

Proof: (a) $\Rightarrow$(b) it is easy to verify.

(b) $\Rightarrow$(c) if every R-module is pure w-projective then $Ext_R^{\,1}(M_1 + M_2, N) = O$

(d) $\Rightarrow$ (a) det M be and pure w-injective R-module. We have the following exact commutative diagram.



Note that $\sigma L\gamma\sigma M = O = O\sigma M$, so $\sigma L\gamma = O$ by (d) therefore $L = im(\gamma) \subseteq \ker(\sigma L) = O$ and hence M is pure w-injective where $M = M_1 + M_2$.

Let l is a class of R-modules and M is an R-module. A homomorphism $\phi \in Hom_R(N, M)$ with $N \in l$ is called and l pure-precover of M if the induces map $HOM_R(1_N, \phi)$**:**

$HOM_R(N^1, N) \to HOM_R(N^1, M_1 + M_2)$ is surjective for all $N^1 \in l$. An l-pure cover $\phi \in HOM_R(N, M + M_2)$ is called an l-pure cover if each $\gamma \in HOM_R(N, M_1 + M_2)$ is called an l-pure cover if each $\gamma \in HOM_R(N, N)$ satisfying $\phi = \phi\gamma$ is an automorphism of N. The class l is called a pure precover class if every R-module has an l-pure precover.

If l is the class of pure w-injective R-modules then an l-envelope is called a pure w-injective envelope.

Proposition : If M is an R-module, then the following are equivalent:

(i) M is pure w-projective; Where $M = M_1 + M_2$

(ii) M is pure projective with respect to every exact sequence $O \to A \to B \to C \to O$, where A is pure w-projective.

(iii) For every exact sequence $O \to K \to F \to M \to O$ with kerf = O where F is pure W- injective, $K \to F$ is a pure w-injective preenvelope of K.

(iv) M is cokernel of a pure w-injective preenvelope $K \to F$ with F projective.

Proof: (i) $\Rightarrow$ (ii) Let $O \to A \to B \to C \to O$ is an exact sequence where A is pure w-injective. Then $Ext_R^1(M_1 + M_2 + A) = O$ by (i), Thus $HOM_R(M_1 + M_2, B) \to HOM_R(M_1 + M_2, C) \to O$ is exact, and (ii) holds.

(ii) $\Rightarrow$ (i) For every pure w-injective R-module N, there is a short exact sequence $O \to N \to E \to L \to O$ with E injective, which induces an exact sequence $HOM_R(M_1 + M_2, E) \to HOM_R(M_1 + M_2, L) \to Ext_R^1(M_1 + M_2, N) \to O$. Since

$HOM_R(M_1 + M_2, E) \to HOM_R(M_1 + M_2, L) \to O$ is exact by (ii), we have $Ext_R^1(M_1 + M_2, N) = O$ and (i) follows.

(i) $\Rightarrow$ (iii) it is easy to verify.

(iii)$\Rightarrow$(iv) Let $O \to K \to P \to M \to O$ is an exact sequence with P-pure projective and $M = M_1 + M_2$. Note P is pure w-injective by hypothesis, thus $K \to P$ is a pure w-injective preenvelope.

(iv) $\Rightarrow$ (i) There is an exact sequence $O \to K \to P \to M \to O$ where $K \to P$ is a pure w-injective preenvelope with P pure projective. It gives rise to the exactness of $HOM_R(P, N) \to HOM_R(K, N) \to Ext_R^1(M, N) \to O$ for each pure w-injective R-module N.

Note that $HOM_R(P, N) \to HOM_R(K, N) \to O$ is exact by (iv). Hence $Ext_R^1(M, N) = O,$ as desired. Where $M = M_1 + M_2$

PURE W-PROJECTIVE DIMENSION OVER SEMI-DEDEKIND

Definition: (a) For any R-module M, let pure w-injective dimension P wid (M) of M, denote the smallest integer $n \geq o$ such that $Ext_R^{n+1}(N, M) = O$ for every R-module N of weak dimension $\leq$ 1. (If no such n exists, set P wid (M) = ∞)

(b) P wid (R) = Sup { P wid (M) : M is an R-module}.

Lemma : Let R be a semi-Dedekind domain. For an R-module M, the following statement are equivalent:

(i) P wid (M) $\leq$ n:

(ii) $Ext_R^{n+1}(N,M) = O$ for all R-modules N of W- dimension $\leq$ 1.

(iii) If the sequence $O \to M \to E_o \to E_1 \to .... \to E_n \to O$ is exact with $E_0, E_1........E_{n-1}$ Pure w-injective, then also E_n is Pure w-injective.

Proof: (i) $\Rightarrow$ (ii) Use induction on n. Clear if P wid (M) = n.If P wid (M) $\leq$ n – 1 resolve N by $O \to K \to P \to N \to O$ with K and P flat. K have P w dimension $\leq$ 1 and $Ext_R^{n+1}(N,M) \cong Ext_R^{n}(K,M) = O$ by induction hypothesis.

(ii) $\Rightarrow$ (iii) follows from the isomorphism $Ext_R^{n+1}(N,M) \cong Ext_R^{1}(N,E_n)$.

(iii) $\Rightarrow$ (i) trivial.

Proposition : Let R be a semi-Dedekind domain. For and R-module M and an integer n $\geq$ o, the following are equivalent:

(i) P wpd M) $\leq$ n;

(ii) $Ext_R^{n+1}(N,M) = O$ for any pure w-injective R-module N;

(iii) $Ext_R^{n+j}(M,N) = O$ for any pure w-injective R-module N and j $\geq$ I.

(iv) There exists an exact sequence $O \to P_n \to P_{n-1} \to .... \to P_1 \to P_o \to M \to O$ where each P_i is pure w-projective.

Proof: (ii) $\Rightarrow$ (iii) For any pure w-injective R-module N, there is a short exact sequence $O \to N \to E \to L \to O$, where E is injective. Then the sequence $Ext_R^{n+1}(N,L) \to Ext_R^{n+2}(M,N) \to Ext_R^{n+2}(M,E) = O$ is exact. Note that L is pure w-injective so $Ext_R^{n+1}(M,N) = O$ by (ii)

Hence $Ext_R^{n+2}(M,N) = O$

(i) $\Rightarrow$ (ii) is similar to (ii) $\Rightarrow$ (iii)

(i) $\Leftrightarrow$ (iv) is straightforward

(ii) $\Rightarrow$ (i) is obvious.

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