SOME CONDITION REGARDING TO COMMUTING AND NON COMMUTING EXPONENTIAL MATRIX

Mohammed Abdullah Salman and V. C. Borkar

Department of Mathematics & Statistics, Yeshwant Mahavidyalaya, Swami Ramamnand Teerth Marthwada University, Nanded, India.

ABSTRACT

Accordingly to the previous paper for authors [1]. The exponential function defined on non-commutative algebra but does not occur in the general form of equation $e^{x+y} = e^x e^y$ *. In this paper we define the conditions for which this equation is valid in* $M(2,R)$ *, it will show it more easily and it shows some science achievements over 50's.*

Keywords: *Matrix Exponential, Commuting Matrix, Non-commuting Matrix*

INTRODUCTION

It is known that in the numerical fields for the exponential function $\exp(x) = e^x$ satisfies the equation of exponential function $e^{x+y} = e^x e^y$. This equality is not true in general cases when the exponential function is defined on the matrices especially when non-commutative of matrices are used. But it is known that the equation is verified if the two exponential of matrices commutative:

$$AB = BA \quad \text{then} \quad e^{A+B} = e^A e^B = e^B e^A .$$

But when the application of the converse is not always true the two matrices that do not commute can apply any of the relations:

$$e^{A+B} = e^A e^B = e^B e^A$$
$$e^{A+B} \neq e^A e^B = e^B e^A$$
$$e^{A+B} = e^A e^B \neq e^B e^A$$
$$e^{A+B} \neq e^A e^B \neq e^B e^A$$

Several studies have tried to determine the characteristics the matrices that do not commute in the exponential of matrices. In particular, the problem has been studied for 50 years for matrices of dimension two or three see for more details [2, 3, 5, 6, 8, and 12] and also taken up recently in [3]. The simplest case is that of matrices in $M(2,R)$, discussed and solved in

[3] Under the more general complex algebra degree two. In this paper it proposed a Simple discussion on how to characterize the matrices $M(2,R)$ for which we have:

$$e^A e^B = e^B e^A = e^{A+B} \qquad (1)$$

and it shows that do not exist matrices for which we have:

$$e^A e^B = e^B e^A \neq e^{A+B}$$

Definitions:

The set of real matrices 2×2, $M(2,R)$ is a vector space over R with respect to the operations of matrix addition and multiplication by a real number, an algebra is not commutative respect of those operations and the usual matrix product, and is a complete metric space from the norm:

$$\|A\| = \sup_{|x|=1} |Ax|$$

In the following we will consider matrices $M(2,R)$ such that a matrix $A = \begin{pmatrix} a & \mathrm{b} \\ c & d \end{pmatrix}$ is invertible (non-singular) if and only if the determination does not equal zero det $A = ad - bc \neq 0$ and its inverse is given by $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} d & -\mathrm{b} \\ -c & \mathrm{a} \end{pmatrix}$

The set of invertible matrices, denoted by $GL(2,R)$ is a non-commutative group under the operation of the product of matrices, whose neutral element is the matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The trace of a matrix is the sum of its elements on the main diagonal:

$$trace\ A = trace\begin{pmatrix} a & \mathrm{b} \\ c & d \end{pmatrix} = a + d$$

The commutator of two matrices A, B is the matrix which defined by:

$$[A, B] = AB - BA$$

If $[A,B] \neq 0$ then A, B and I are linearly independent.

The centre $C(2,R)$ of $M(2,R)$ is the set of matrices $\mathrm{X} \in M(2,R)$ that commute at all Matrices $M(2,R)$:

$$C(2,R) = \{X : [A, X] = 0, \forall A \in M(2,R)\}$$

and is the subgroup of $GL(2,R)$ constituted by the scalar matrices: $X = xI$ with $\mathrm{x} \in \mathrm{R}$. $C(2,R)$ is a Lie group and is obviously isomorphic to R^*. The sign of $\mathrm{X} \in C(2,R)$ is the sign of $\mathrm{x} \in \mathrm{R}$.

Exponential of a matrix:

The exponential of a matrix is defined by:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \ldots\ldots + \frac{A^{n-1}}{(n-1)!} + \ldots\ldots.. \qquad (`2)$$

The series (2) is absolutely convergent and defines an entire function in C, so it is convergent in the metric space $M(2,R)$. Since the product of matrices in $M(2,R)$ is not commutative, the exponential function so defined does not satisfy, in general, the equation (1). However, apply the following properties which we will use in the following:

$$e^{A}e^{-A} = e^{0} = I$$

If A is invertible, then $\det A \neq 0 \Rightarrow e^{ABA^{-1}} = Ae^{B}A^{-1}$

$$AB = BA \Rightarrow \; e^{At}e^{Bt} = e^{Bt}e^{At} = e^{(B+A)t} \quad \forall t \in C$$

For more details and examples of these properties see [1,2, 3, 6, 7, 8, 9, 10,11].

Calculation of the exponential of a matrix:

The calculation of the matrix exponential $M(n,K)$ quickly becomes very complex as n increases. However, there are procedures that allow always make such a calculation in a finite number of steps, at least in principle [6, 7, 9, 12, 13]. In the case of matrices $M(2,R)$ the calculation of the exponential is quite simple with the following decomposition.

Lemma (1)

Each matrix $A \in M(2,R)$ can be decomposed into a sum of two matrices one of which is in the centre $C(2,R)$ and the other has null trace [1]:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = kI + A' = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} m & b \\ c & -m \end{pmatrix}$$

And, for two matrices $A, B \in M(2,R)$, we have:

$$[A', B'] = 0 \Leftrightarrow [A, B] = 0.$$

Lemma (2)

If M is a traceless matrix [1], where $\theta = \sqrt{\det M}$ we have [1]:

$$e^{M} = I\cos\theta + M\frac{\sin\theta}{\theta}$$

Theorem (1)

For each matrix $A \in M(2,R)$, we have [1]:

$$e^{A} = e^{kI+A'} = e^{kI}e^{A'} = e^{k}(I\cos\alpha + A'\frac{\sin\alpha}{\alpha})$$

With,

$$k = \frac{trace\ A}{2}, A' = A - kI, \; \alpha = \sqrt{\det A'}$$

Lemma 3

Given a nonzero matrix [1]:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, e^{A} \in C(2,R)$$

If and only if $A \in C(2,R)$, $\det A' = \det A - (\frac{trace\ A}{2})^2 = \mu^2\pi^2$ with $\mu \in N^+$.

MAIN RESULTS

Let's start with finding the necessary and sufficient conditions for the exponential of two commute matrices. We know that this is true if the two matrices commute, so we are only interested in matrices such that $[A,B] \neq 0$, and of course a sufficient condition is that at least one of the two exponential is in the center $C(2,R)$. In the case of matrices in $M(2,R)$ this condition is also necessary and their some conditions see [4]

Theorem (2)

Given two matrices A,B such that

$$[A,B] \neq 0, e^A e^B = e^B e^A \text{ if and only if } e^A \in C(2,R) \text{ or } e^B \in C(2,R)$$

Proof:

We have to show only the $\{e^A e^B = e^B e^A\} \Rightarrow \{e^A \in C(2,R) \text{ or } e^B \in C(2,R)$.

Using Lemmas (1) and (2) we have:

$$e^A = e^k(I\cos\alpha + A'\frac{\sin\alpha}{\alpha}), \qquad e^B = e^h(I\cos\beta + B'\frac{\sin\beta}{\beta})$$

Calculating the commutator and noticing that $I\cos\alpha$ and $I\cos\beta$ are elements in $C(2,R)$

Then we have:

$$[e^A, e^B] = e^{k+h}[I\cos\alpha + A'\frac{\sin\alpha}{\alpha}, I\cos\beta + B'\frac{\sin\beta}{\beta}] = e^{k+h}\frac{\sin\alpha\sin\beta}{\alpha\beta}[A', B']$$

And since, $[A',B'] = [A,B] \neq 0$, exponentials commute only if :

$$\sin\alpha\sin\beta = 0 \text{ with } \alpha,\beta \neq 0, \text{ and then } \alpha = \mu\pi \text{ or } \beta = \phi\pi \text{ with } \mu,\phi \in N^+.$$

The fact that the commute exponentials does not be valid when it applies the equation (1), even if e^A, e^B are in the center $C(2,R)$. For example, consider two matrices of the type:

$$A = \pi\begin{pmatrix} a & -\mu \\ \mu & a \end{pmatrix}, \quad B = \pi\begin{pmatrix} b+\phi c & -\phi - c^2 \\ \phi^2 & b-\phi c \end{pmatrix}$$

If $\mu \in N^+$ then $e^A = (-1)^{\mu} e^{a\pi} I$ and if $\phi \in N^+$ then $e^B = (-1)^{\phi} e^{b\pi} I$,then:

$$e^A e^B = e^B e^A = (-1)^{\mu+\phi} e^{(a+b)\pi} I$$

But it has $e^{A+B} = e^A e^B$ only if:

$$c^2 = \frac{\nu - (\mu + \phi^2)^2}{\mu}$$

With ν full equal to $\mu+\phi$.

The conditions for which the equation (1) is verified are given by the following theorem.

Theorem (3)

Given two matrices $A,B\in M(2,R)$, $e^{A+B}=e^{A}e^{B}=e^{B}e^{A}$.if and only if commuting or e^{A+B},e^{A},e^{B} they are in the center $C(2,R)$ and the sign e^{A+B} is equal to the sign of $e^{A}e^{B}$.

Proof

We have to show only the case in which A and B not commuting. In case $\alpha=\mu\pi$ with $\mu\in\mathrm{R}^{+}$ we know that e^{A} and e^{B} commuting. We are looking for the conditions for which it has also $e^{A+B}=e^{A}e^{B}$ commuting.

Let us assume:

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}=kI+A' \qquad \text{with :} \qquad A'=\begin{pmatrix} \mathrm{m} & \mathrm{b}\\ \mathrm{c} & -\mathrm{m}\end{pmatrix},\ k=\frac{a+d}{2},\ m=\frac{a-d}{2}$$

$$B=\begin{pmatrix} \mathrm{x} & y\\ z & t\end{pmatrix}=hI+B' \qquad \text{with :} \qquad B'=\begin{pmatrix} \mathrm{n} & \mathrm{y}\\ \mathrm{z} & -\mathrm{n}\end{pmatrix},\ h=\frac{x+t}{2},\ n=\frac{x-t}{2}$$

It has therefore :

$$e^{A}=e^{k}(I\cos\alpha+A'\frac{\sin\alpha}{\alpha}) \qquad \text{with :} \qquad \alpha=\sqrt{\det(\mathrm{A}')}=\sqrt{-m^{2}-bc}$$

$$e^{B}=e^{h}(I\cos\beta+B'\frac{\sin\beta}{\beta}) \qquad \text{with :} \qquad \beta=\sqrt{\det(\mathrm{B}')}=\sqrt{-n^{2}-yz}$$

We get

$$e^{A}e^{B}=e^{k+h}(I\cos\alpha\cos\beta+B'\frac{\cos\alpha\sin\beta}{\beta}+A'\frac{\cos\beta\sin\alpha}{\alpha}+A'B'\frac{\sin\alpha\sin\beta}{\alpha\beta}) \tag{3}$$

And put $\alpha=\mu\pi$ we have $e^{A}=(-1)^{\mu}e^{k}I$ and the (3) become:

$$e^{A}e^{B}=e^{k+h}(-1)^{\mu}\,(I\cos\beta+B'\frac{\sin\beta}{\beta})=$$

$$=e^{k+h}(-1)^{\mu}\left[\cos\beta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\frac{\sin\beta}{\beta}\begin{pmatrix} n & y\\ z & -n\end{pmatrix}\right]=$$

$$=e^{h+k}(-1)^{\mu}\begin{pmatrix}\cos\beta+\dfrac{n\sin\beta}{\beta} & \dfrac{y\sin\beta}{\beta}\\ \dfrac{z\sin\beta}{\beta} & \cos\beta-\dfrac{n\sin\beta}{\beta}\end{pmatrix}$$

On the other hand, we have:

$$C = A + B = \begin{pmatrix} a+x & b+y \\ c+z & d+t \end{pmatrix} = (h+k)I + C'$$

With:

$$C' = \begin{pmatrix} m+n & b+y \\ c+z & -(m+n) \end{pmatrix} = A' + B'$$

and put

$$\gamma = \sqrt{\det(C')} = \sqrt{-q^2 - (b+y)(c+z)}$$

Then we obtained

$$e^C = e^{h+k}\left(I\cos\gamma + C'\frac{\sin\gamma}{\gamma}\right) = e^{h+k}\begin{pmatrix} \cos\gamma + \frac{(m+n)\sin\gamma}{\gamma} & \frac{(b+y)\sin\gamma}{\gamma} \\ \frac{(c+z)\sin\gamma}{\gamma} & \cos\gamma - \frac{(m+n)\sin\gamma}{\gamma} \end{pmatrix}$$

Equating the corresponding terms of the two matrices we have:

$$\begin{cases} (b+y)\frac{\sin\gamma}{\gamma} = y\frac{\sin\beta}{\beta}(-1)^\mu \\ (c+z)\frac{\sin\gamma}{\gamma} = z\frac{\sin\beta}{\beta}(-1)^\mu \\ (m+n)\frac{\sin\gamma}{\gamma} = (-1)^\mu(\cos\beta + \frac{n\sin\beta}{\beta} - \cos\gamma) \\ -(m+n)\frac{\sin\gamma}{\gamma} = (-1)^\mu(\cos\beta - \frac{n\sin\beta}{\beta} - \cos\gamma) \end{cases} \quad (4)$$

Now we have two possible cases:

Case (1):

If $\frac{\sin\gamma}{\gamma} \neq 0$, in this case, from the last two equations of the system (4) we have

$$\begin{cases} m = \frac{\gamma}{\sin\gamma}[(-1)^\mu\cos\beta - \cos\gamma] + n\left[(-1)^\mu\frac{\gamma\sin\beta}{\beta\sin\gamma} - 1\right] \\ m = -\frac{\gamma}{\sin\gamma}[(-1)^\mu\cos\beta - \cos\gamma] + n\left[(-1)^\mu\frac{\gamma\sin\beta}{\beta\sin\gamma} - 1\right] \end{cases}$$

This yield:

$$(-1)^\mu\cos\beta - \cos\gamma = -[(-1)^\mu\cos\beta - \cos\gamma]$$

That occurred when:

$$\gamma = \pm\beta + 2\nu\pi \text{ for } \mu \text{ even }, \qquad \gamma = \pm\beta + (2\nu+1)\pi \text{ for } \mu \text{ odd}$$

And in these cases we have:

$$m = n\left(\frac{(-1)^{\mu}\gamma\sin\beta}{\beta\sin\gamma} - 1\right) = \omega n$$

And from the other two equations of the system (4) we are obtained:

$$b = y\left(\frac{(-1)^{\mu}\gamma\sin\beta}{\beta\sin\gamma} - 1\right) = \omega y$$

$$c = z\left(\frac{(-1)^{\mu}\gamma\sin\beta}{\beta\sin\gamma} - 1\right) = \omega z$$

Then we have:

$$A' = \omega B'$$

So that the matrices A' and B', then also A and B are commute.

Case (2):

If $\sin\gamma = 0$ with $\gamma \neq 0$, where $\gamma = \nu\pi$ with $\nu \in N^{+}$. Then we have

$$e^{C} = (-1)^{\nu} e^{k+h} I$$

And to have $e^{C} = e^{A}e^{B}$ it must be:

$$\sin\beta = 0 \text{ and } (-1)^{\nu} = (-1)^{\mu}\cos\beta$$

Then:

$$\beta = \phi\pi \text{ with } (-1)^{\phi+\mu} = (-1)^{\nu}.$$

In this case all matrices $e^{A}, e^{B}, e^{A}e^{B}$, e^{A+B} are in the center $C(2,R)$ and has the same equality of

$\mu+\phi$ and then e^{A+B} has the same sign as $e^{A}e^{B}$. Note that if ν has equal opposite $\mu+\phi$ then we have:

$$e^{A+B} = -e^{A}e^{B}.$$

It remains to verify the possibility that the exponential of the sum equals to only one of the two products of exponentials, but this situation is impossible in $M(2,R)$, as the following theorem.

Theorem (4)

For two matrices $A, B \in M(2,R)$ is impossible to have

$$e^{A+B} = e^{A}e^{B} \neq e^{B}e^{A}$$

Proof:
Obviously sufficient to show that it is impossible to have

$$e^{A'+B'} = e^{A'}e^{B'} \neq e^{B'}e^{A'}$$

Equating $e^{A'}e^{B'} = e^{A'+B'}$ we have

$$I\cos\alpha\cos\beta + A'\frac{\sin\alpha\cos\beta}{\alpha} + B'\frac{\cos\alpha\sin\beta}{\beta} + A'B'\frac{\sin\alpha\sin\beta}{\alpha\beta} = I\cos\gamma + A'\frac{\sin\gamma}{\gamma} + B'\frac{\sin\gamma}{\gamma} \qquad (5)$$

by theorem (2) we know that if A', B' non-commute then $e^{A'}e^{B'} \neq e^{B'}e^{A'}$. Equivalent a $\sin\alpha\sin\beta \neq 0$ with $\alpha, \beta \neq 0$. In that case $A'B'$ we can then express as a linear combination of the three linearly independent matrices A', B', I.

$$A'B' = \frac{\alpha\beta}{\sin\alpha\sin\beta}\left[I(\cos\gamma - \cos\alpha\cos\beta) + A'\left(\frac{\sin\gamma}{\gamma} - \frac{\sin\alpha\cos\beta}{\alpha}\right) + B'\left(\frac{\sin\gamma}{\gamma} - \frac{\cos\alpha\sin\beta}{\beta}\right)\right] =$$
$$= \xi_1 I + \xi_2 A' + \xi_3 B' \qquad \text{with } \xi_1, \xi_2, \xi_3 \in R$$

Since A' and B' elements of algebra $M(2, R)$ it must have:

$$(A')^2 B' = A'(A'B')$$

Therefore:

$$-\alpha^2 B' = (\xi_1\xi_3 - \xi_2\alpha^2)I + (\xi_1 + \xi_2\xi_3)A' + \xi_3^{\,2} B'$$

and for the linear independence it is obtained

$$\xi_3^{\,2} = -\alpha^2, \xi_1 + \xi_2\xi_3 = 0$$

for the same reason it has also:

$$(A')B'^2 = (A'B')B'$$

which yields the same way:

$$-\beta^2 = \xi_2^{\,2}$$

and since ξ_1, ξ_2, ξ_3 must be real numbers, if this is impossible α and β are real numbers, namely $\det A'$ and $\det B'$ are positive.
Let us consider only the case in which $\det A'$ and $\det B'$ are negative, then:
$\alpha = i|\alpha|$ and $\beta = \mathrm{i}|\beta|$, for which we have: $\xi_2 = i\beta$, $\xi_3 = i\alpha$ and $\xi_1 = -\xi_2\xi_3 = \alpha\beta = -|\alpha||\beta|$
And then:

$$A'B' = \alpha\beta I + i\beta A' + i\alpha B'$$

for which is obtained:

$$e^{A'}e^{B'} = \frac{\sin\alpha\sin\beta}{\alpha\beta}(\alpha\beta\, I + i\beta\, A' + i\alpha\, B') + \frac{\sin\alpha\cos\beta}{\alpha}A' + \frac{\cos\alpha\sin\beta}{\beta}B' + \cos\alpha\cos\beta\, I$$

$$= I(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + A'\left(\frac{\sin\alpha\cos\beta}{\alpha} + i\frac{\sin\alpha\sin\beta}{\alpha}\right) + B'\left(\frac{\sin\beta\cos\alpha}{\beta} + i\frac{\sin\alpha\sin\beta}{\beta}\right)$$

Then we have:

$$e^{A'}e^{B'} = e^{A'+B'} = \cos\gamma\, \mathrm{I} + \frac{\sin\gamma}{\gamma}A' + \frac{\sin\gamma}{\gamma}B'$$

Then We get :

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos\gamma$$

This is occurred when $\gamma = \alpha - \beta$.

From the other two components we are obtained:

$$\frac{\sin\alpha\cos\beta}{\alpha} + i\frac{\sin\alpha\sin\beta}{\alpha} = \frac{\sin\beta\cos\alpha}{\beta} + i\frac{\sin\alpha\sin\beta}{\beta}$$

Where α and β are imaginary numbers we rewrite this equality by using hyperbolic functions that we mention it in the notation of theorem (1), and we get:

$$\frac{e^{-|\beta|}(e^{|\alpha|} - e^{-|\alpha|})}{2|\alpha|} = \frac{e^{-|\alpha|}(e^{|\beta|} - e^{-|\beta|})}{2|\beta|}$$

Then we get:

$$\frac{e^{2|\alpha|} - 1}{2|\alpha|} = \frac{e^{2|\beta|} - 1}{2|\beta|}$$

but this is impossible for two different real numbers because the function:

$$y = \frac{e^{x} - 1}{x}$$

It is monotonically increasing for $x > 0$.

CONSTRUCTION OF MATRICES THAT VERIFY THE EQUATION

We want now to define a procedure to construct two matrices verifies the equation (1). Let's start with the construction of two traceless matrices, that do not commute, and such that the square root of their determinant is a positive integer multiple of π. A matrix that satisfies their conditions has the form:

$$A_o = \mu\pi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mu \in N^{+}$$

a matrix that does not commute with A_o and that verification of the same conditions must have the form:

$$B_o = \psi\pi \begin{pmatrix} x & -y \\ \frac{1+x^2}{y} & -x \end{pmatrix} \qquad \psi \in N^+ \; x, y \neq 0$$

For these two matrices we have:

$$e^{A_o} = (-1)^{\mu} I \qquad e^{B_o} = (-1)^{\psi} I \qquad e^{A_o} e^{B_o} = (-1)^{\mu+\psi} e^{I}$$

Then we have:

$$A_o + B_o = (\mu+\psi)\pi \begin{pmatrix} x & -1-y \\ \frac{1+(1+x^2)}{y} & -x \end{pmatrix}$$

and then:

$$\det(A_o + B_o) = \left[-x^2 + (1+y\frac{1+y+x^2}{y}) \right] (\mu+\psi)^2 \pi^2$$

then to verify the equation (1) must be:

$$\frac{x^2}{y} + \frac{(1+y)^2}{y} = 4\nu^2 \qquad \nu \in \mathrm{N}^+$$

that is:

$$x = \sqrt{4\nu^2 y - (1+y)^2}$$

then:

$$B_o = \psi\pi \begin{pmatrix} \sqrt{4\nu^2 y - (1+y)^2} & -y \\ 4\nu^2 - y - 2 & -\sqrt{4\nu^2 y - (1+y)^2} \end{pmatrix} \qquad \psi, \nu \in N^+$$

and in case one has:

$$\det(A_o + B_o) = 4\nu^2 (\mu+\psi)^2 \pi^2$$

then:

$$e^{A_o + B_o} = (-1)^{2\nu(\mu+\psi)} I = (-1)^{\mu+\psi} I = e^{A_o} e^{B_o}$$

Recall now that similar matrices have the same determinant, then a date of a nonsingular matrix p can build two similar matrices to A_o and B_o such that:

$$A' = P^{-1} A_o P \qquad B' = P^{-1} B_o P$$

that still do not commute and being:

$$[A', B'] = P^{-1} [A_o, B_o] P$$

and finally two matrices A, B:

$$A = kI + A' \qquad B = hI + B'$$

such that :

$$e^{A+B} = e^{A}e^{B} = .e^{B}e^{A} = (-1)^{\mu+\psi} e^{k+h} I$$

CONCLUSION

The problem of determining the necessary and sufficient conditions for two matrices $M(2,R)$ Verify the equation $e^{A+B} = e^{A}e^{B} = e^{B}e^{A}$ is thus entirely solved. The result of Theorem (3) can also be expressed in topological terms, noting that the center $C(2,R)$ is a Lie group is not connected, obviously isomorphic to the multiplicative group $\{R^{+}, X\}$ and its two components are connected sets of matrices scalar positive and negative. Theorem (3) can then be reformulated saying that $e^{A+B} = e^{A}e^{B}$ if they belong to the same connected component of $C(2,R)$. It is not difficult to extend these results to the case of two matrices $M(2,R)$, just note that the its center $C(2,R)$ is connected, then in Theorem (3) just ask that e^{A+B}, $e^{A}e^{B}$, $e^{B}e^{A}$ are in the center, while Theorem (4) is not longer valid because the function $w = (e^{z} - 1)/z$ periodic in C and therefore $M(2,C)$ there are infinite matrices such that $e^{A+B} = e^{A}e^{B} \neq e^{B}e^{A}$.From a first search in the literature it does not seem to be a solution in the general case of matrices $M(n,K)$ with $K = R$ or C, even in the case n = 3.

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